On Treewidth, Separators and Yao's Garbling

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Our Result

- **Theorem.** For Boolean circuits of size S and *treewidth* $w = w(S)$, Yao's garbling Γ is *adaptively-indistinguishable* with a loss in security $S^{O(w)}$.

- **Remarks:**
  1. Applebaum et al. [AIKW13] ruled out adaptive-*simulatability* of Γ
  2. Jafargholi-Wichs [JW16] proved adaptive-simulatability of Γ', a *variant* of Γ
  3. We can prove adaptive-simulatability of Γ' in terms of treewidth

Overview

Garbling

Security Models

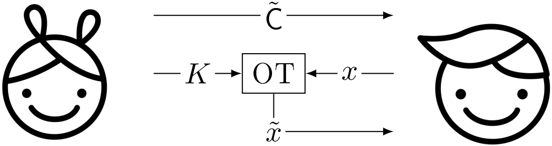
Yao's Garbling

Our Reduction

Garbling [Y86]

Circuit $\mathsf{C} : \{0,1\}^n \to \{0,1\}^\ell$

Input $x \in \{0,1\}^n$



$\tilde{\mathsf{C}}$

K

OT

x

$\tilde{x}$

$(\tilde{\mathsf{C}}, K) \leftarrow \mathsf{GCircuit}(\mathsf{C}, 1^\lambda)$

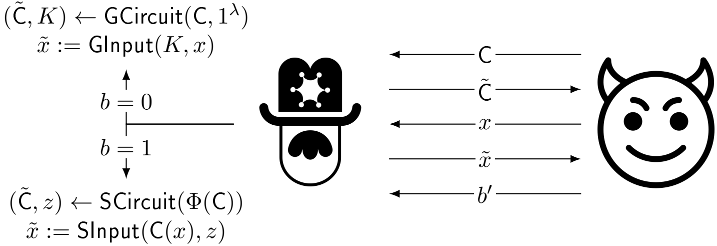
$\mathsf{C}(x) := \mathsf{GEval}(\tilde{\mathsf{C}}, \tilde{x})$

Garbling [BHR12]

- **Syntax**
  - $(\tilde{\mathsf{C}}, K) \leftarrow \mathsf{GCircuit}(\mathsf{C}, 1^\lambda)$
  - $\tilde{x} \leftarrow \mathsf{GInput}(x, K)$
  - $y := \mathsf{GEval}(\tilde{\mathsf{C}}, \tilde{x})$
- **Correctness** $\forall\lambda$, $\forall\mathsf{C}, \forall x$:

$$\Pr_{\substack{(\tilde{\mathsf{C}},K)\leftarrow\mathsf{GCircuit}(\mathsf{C},1^\lambda)\\ \tilde{x}\leftarrow\mathsf{GInput}(x,K)}}\left[\mathsf{GEval}(\tilde{\mathsf{C}}, \tilde{x}) = \mathsf{C}(x)\right] = 1$$

Security: Adaptive Simulatability



$(\tilde{\mathsf{C}}, K) \leftarrow \mathsf{GCircuit}(\mathsf{C}, 1^{\lambda})$

$\tilde{x} := \mathsf{GInput}(K, x)$

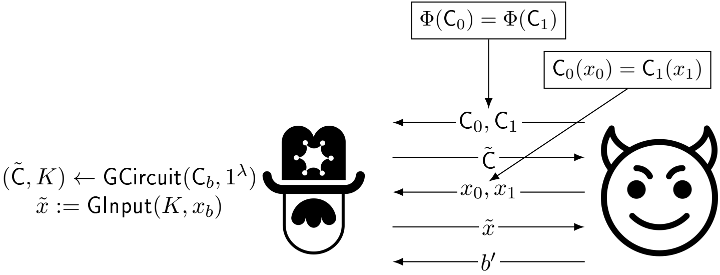
$b = 0$

$b = 1$

$(\tilde{\mathsf{C}}, z) \leftarrow \mathsf{SCircuit}(\Phi(\mathsf{C}))$

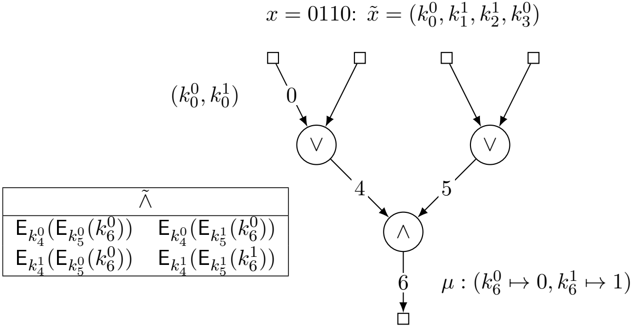
$\tilde{x} := \mathsf{SInput}(\mathsf{C}(x), z)$

Security: Adaptive Indistinguishability


$\Phi(\mathsf{C}_0) = \Phi(\mathsf{C}_1)$
$\mathsf{C}_0(x_0) = \mathsf{C}_1(x_1)$
$(\tilde{\mathsf{C}}, K) \leftarrow \mathsf{GCircuit}(\mathsf{C}_b, 1^\lambda)$
$\tilde{x} := \mathsf{GInput}(K, x_b)$
$\mathsf{C}_0, \mathsf{C}_1$
$\tilde{\mathsf{C}}$
x_0, x_1
$\tilde{x}$
b'

- Adaptive Simulatability $\implies$ Adaptive Indistinguishability
- Application: restricted symmetric-key FE [JSW17]

Yao's Garbling Γ



$x = 0110$: $\tilde{x} = (k_0^0, k_1^1, k_2^1, k_3^0)$

$\tilde{\wedge}$	
$\mathsf{E}_{k_4^0}(\mathsf{E}_{k_5^0}(k_6^0))$	$\mathsf{E}_{k_4^0}(\mathsf{E}_{k_5^1}(k_6^0))$
$\mathsf{E}_{k_4^1}(\mathsf{E}_{k_5^0}(k_6^0))$	$\mathsf{E}_{k_4^1}(\mathsf{E}_{k_5^1}(k_6^1))$

- Each wire in $w \in \mathsf{C}$ associated with secret keys (k_w^0, k_w^1)
- Garbled circuit, $\tilde{\mathsf{C}} := (\{\tilde{g}\}_{g \in \mathsf{C}}, \mu)$
    - Garbling table: $\tilde{g}$ for each gate $g \in \mathsf{C}$
    - Output map, μ: (k_w^0, k_w^1) of each o/p wire w mapped to bit
- Garbled i/p, $\tilde{x}$: keys of the i/p wires *selected* using x
- Evaluate: evaluate C "over the encryption"

YAO'S GARBLING Γ...

- Γ: *Online-complexity* depends only on $|x| = n$ (and security parameter)

- **Variant** Γ': o/p map μ sent in *online* phase
  - Garbled circuit: $\tilde{\mathsf{C}} := \{\tilde{g}\}_{g \in \mathsf{C}}$:
  - Garbled i/p: $(\tilde{x}, \mu)$

  - Online complexity depends also on the o/p length ℓ

- E.g.: garbling of a PRG $\mathsf{C} : \{0,1\}^n \to \{0,1\}^{n^c}$
  - Online complexity using Γ' is $\approx n^c$
  - Cannot be adaptively simulatable using Γ

Yao's Garbling: Security Landscape

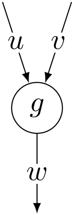
	Selective		Adaptive	
	Γ	Γ'	Γ	Γ'
Simulatability	[LP09]		[AIKW13]	[JW16]
Indistinguishability			This work	

Our Reduction

Garbling Modes

REAL	
$\mathsf{E}_{k_u^0}(\mathsf{E}_{k_v^0}(k_w^{g(0,0)}))$	$\mathsf{E}_{k_u^0}(\mathsf{E}_{k_v^1}(k_w^{g(0,1)}))$
$\mathsf{E}_{k_u^1}(\mathsf{E}_{k_v^0}(k_w^{g(1,0)}))$	$\mathsf{E}_{k_u^1}(\mathsf{E}_{k_v^1}(k_w^{g(1,1)}))$

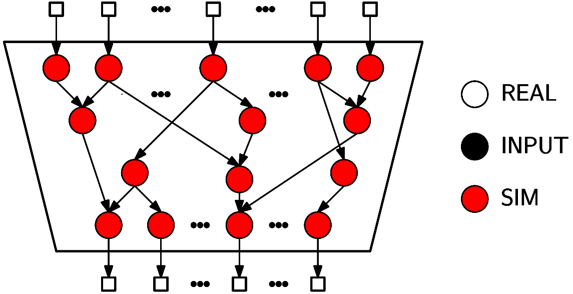
INPUT	
$\mathsf{E}_{k_u^0}(\mathsf{E}_{k_v^0}(k_w^{V(w)}))$	$\mathsf{E}_{k_u^0}(\mathsf{E}_{k_v^1}(k_w^{V(w)}))$
$\mathsf{E}_{k_u^1}(\mathsf{E}_{k_v^0}(k_w^{V(w)}))$	$\mathsf{E}_{k_u^1}(\mathsf{E}_{k_v^1}(k_w^{V(w)}))$



SIM	
$\mathsf{E}_{k_u^0}(\mathsf{E}_{k_v^0}(k_w^{0}))$	$\mathsf{E}_{k_u^0}(\mathsf{E}_{k_v^1}(k_w^{0}))$
$\mathsf{E}_{k_u^1}(\mathsf{E}_{k_v^0}(k_w^{0}))$	$\mathsf{E}_{k_u^1}(\mathsf{E}_{k_v^1}(k_w^{0}))$

- $V(w)$: value of the wire when evaluating $\mathsf{C}(x)$
- Indistinguishability game: $\mathsf{REAL}_0/\mathsf{REAL}_1$, $\mathsf{INPUT}_0/\mathsf{INPUT}_1$

Selective Simulatability [LP09]



- **Hybrid argument**
  1. Replace REAL with INPUT in *topological order*
     - Indistinguishable by ciphertext indistinguishability of SKE
  2. Replace INPUT with SIM in *reverse* topological order:
     - Indistinguishable information-theoretically
- **Programming**
  1. Program o/p map μ so that keys of output wires correctly map to $\mathsf{C}(x)$
- Implies adaptive simulatability with additional 2^n loss

Hurdles to Adaptive Indistinguishability

1. Problem: Input x only available in online phase

   1.1 Problem: Cannot program μ in the offline phase

   - [JW16] solution: Send μ in *online* phase (i.e., Γ'), *defer* programming
   - Our solution: Avoid SIM mode in the hybrids

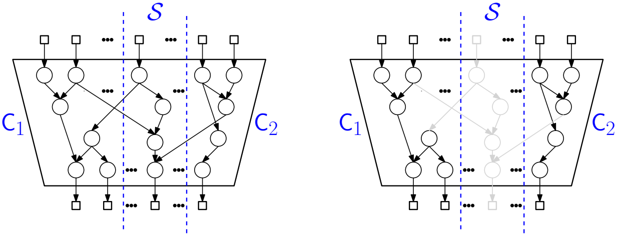
   1.2 Problem: How to simulate INPUT?

   - [JW16] solution: Minimise #INPUT gates and *guess* their values!

2. Problem: How to minimise #INPUT?
   - [JW16] solution: Restrict circuit classes, e.g., low-depth circuits
   - Our solution: *Divide and conquer* via treewidth/separator

Treewidth/Separator

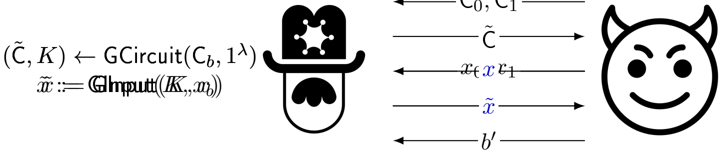
- **Treewidth.** Measure of how 'far' a circuit (DAG) is from a formula (tree)
    - E.g., Boolean formulae have treewidth 1
- **Separator.** A sub-set of gates $\mathcal{S}$ of a circuit C such that *removing* $\mathcal{S}$ (and its incident wires) from C results in *disconnected* sub-circuits of size at most $2/3|\mathsf{C}|$



- **Treewidth-Separator Theorem [RS86].** Any circuit of size S with treewidth $w = w(S)$ has a separator of size w.

Security: Adaptive Indistinguishability

- Simpler indistinguishability game with *single* i/p



$(\tilde{\mathsf{C}}, K) \leftarrow \mathsf{GCircuit}(\mathsf{C}_b, 1^\lambda)$

$\tilde{x} := \mathsf{GInput}(K, x_b)$

$\tilde{x} := \mathsf{GInput}(K, x)$

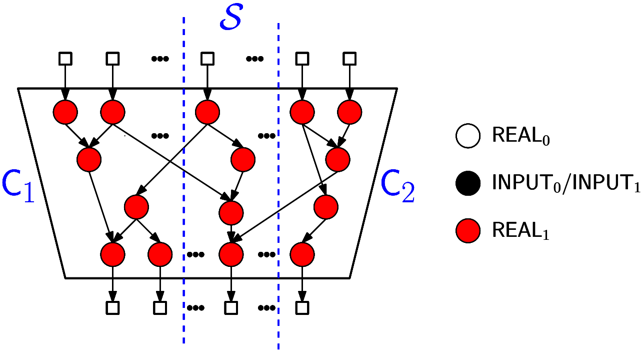
- Garbling modes: $\mathsf{REAL}_0/\mathsf{REAL}_1$, $\mathsf{INPUT}_0/\mathsf{INPUT}_1$

Our Reduction

- **Goal**: Switch all garbling tables from REAL_0 to REAL_1
- **Constraint**: Minimise $\#\mathsf{INPUT}_0/\mathsf{INPUT}_1$ garbling tables
- **Idea**: Maintain $\mathsf{INPUT}_0/\mathsf{INPUT}_1$ gates *only* "at" separator
  - Property of separator $\implies$ can *recurse* on components
  - *Small* separator $\implies$ few $\mathsf{INPUT}_0/\mathsf{INPUT}_1$ gates

Our Reduction...

- Recursive structure of hybrids:
  - Switch gates “on” separator $\mathcal{S}$ to $\mathsf{INPUT}_0/\mathsf{INPUT}_1$
  - *Recursively* switch C_1, C_2 from REAL_0 to REAL_1
  - Switch gates on separator to REAL_1



- $\#\mathsf{INPUT}_0/\mathsf{INPUT}_1 \approx |\mathcal{S}|\delta\log(S)$, δ is the degree

Our Reduction...

- Abstracted out, formalised using a pebble game

- **Lemma** 1. Hybrids corresponding to neighbouring pebble configurations are indistinguishable.
    - Based on ciphertext indistinguishability of SKE or information-theoretically
- **Lemma** 2. There exists a pebble strategy which uses $w\delta \log(S)$ black/gray pebbles.

- **Theorem.** For Boolean circuits of size S and *treewidth* $w = w(S)$, Yao's garbling Γ is *adaptively-indistinguishable* with a loss in security $S^{O(w)}$.
    - Using piecewise-guessing framework [JKK+17]

Thank you!

References

AIKW13 Applebaum, Ishai, Kushilevitz and Waters *Encoding Functions with Constant Online Rate, or How to Compress Garbling Keys*, Crypto 2013

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  1. Sheriff by Oksana Latysheva
  2. Man, Girl by Zuzanna Nebes
  3. Devil by Alina Oleynik